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Nonlinear stability in magnetic fluids of cylindrical interface with mass and heat transfer

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Abstract. The nonlinear Rayleigh-Taylor stability of the cylindrical interface between the vapor and liquid phases of a magnetic fluid is studied when the phases are enclosed between two cylindrical surfaces coaxial with the interface, and when there is mass and heat transfer across the interface. The method of multiple scale expansion is used for the investigation. The evolution of amplitude is shown be governed by a nonlinear Ginzburg-Landau equation. The various stability criteria is discussed, and the region of stability is displayed graphically.

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1 Introduction

The problem of stability of liquids when there is mass and heat transfer across the interface has been investigated by several researchers [1–7]. Hsieh [2] established a general formulation of interfacial flow problem with mass and heat transfer and applied it to the Rayleigh-Taylor and Kelvin-Helmholtz instability problems in plane geometry. In the nuclear reactor cooling of fuel rods by liquid coolants, the geometry of the system in many cases is cylindrical.

Nayak and Chakraborty [3] studied the Kelvin-Helmholtz stability of the cylindrical interface between the vapor and liquid phases of a fluid, when there is a mass and heat transfer across the interface. On the other hand, Elhefnawy [4] studied the effect of a periodic radial magnetic field on the Kelvin-Helmholtz stability of the cylindrical interface between two magnetic fluids when there is mass and heat transfer across the interface. The analysis of these studies was confined within the framework of the linear theory.

The effect of mass and heat transfer across the interface should be taken into account in stability discussions, when the situations are like film boiling of fluids. However, with the linear analysis, the stability criteria remain the same as in the case with the neglect of heat and mass transfer across the interface. Hsieh [5] found that when the vapor region is hotter than the liquid region, as is usually so, the effect of mass and heat transfer tends to inhibit the growth of the instability. Thus, it is clear that such a uniform model based on the linear theory is inadequate to explain the mechanism involved, and the nonlinear theory is needed to reveal the effect of heat and mass transfer on the stability of the system. This problem is of fundamental importance in number of applications such as design of many types of contacting equipment, *e.g.*, boilers, condensers, reactors and others in industrial and environmental processes.

The purpose of this paper is to investigate the nonlinear stability of cylindrical interface between the vapor and liquid phases of a magnetic fluid when there is a mass and heat transfer across the interface. The basic equations with the accompanying boundary conditions are given in Section 2. The first order theory and the linear dispersion relation are obtained in Section 3. In Sections 4 and 5, we have derived second and third order solutions. In Sections 6 and 7, some numerical examples are presented in graphical forms.

2 Formulation of the problem and basic equations

We shall use a cylindrical system of coordinates (r, θ, z) so that in the equilibrium state z-axis is the axis of symmetry of the system. The central solid core has a radius $a^{(1)}$. In the equilibrium state the fluid phase "1", of density $\rho^{(1)}$, and magnetic permeability μ_1 occupies the region $a^{(1)} < r < R$, and, the fluid phase "2", of density $\rho^{(2)}$, and magnetic permeability μ_2 occupies the region $R < r < a^{(2)}$. The temperature at $r = a^{(1)}, r = R$, and $r = a^{(2)}$ are taken as T_1, T_0, and T_2 respectively. The bounding surfaces $r = a^{(1)}$, and $r = a^{(2)}$ are taken as rigid (see Fig. 1). The interface, after a disturbance, is given by the equation

$$F(r, z, t) = r - R - \eta = 0, \tag{1}$$

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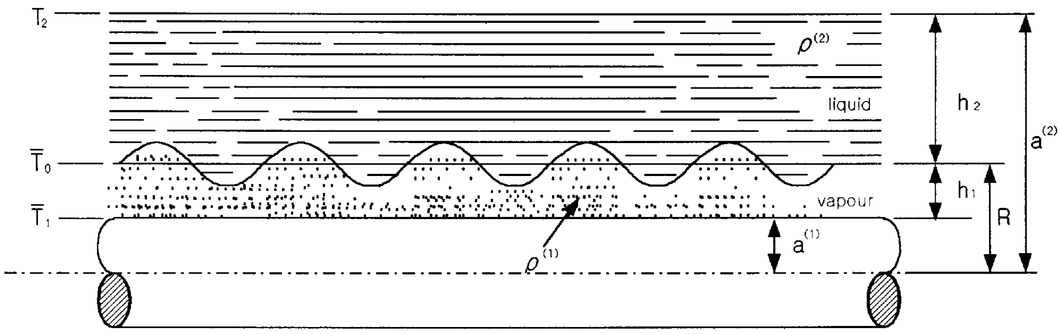

Fig. 1. Configuration under consideration in film boiling.

where η is the perturbation in radius of the interface from its equilibrium value R, and for which the outward unit normal vector is written as

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \left\{1 + \left(\frac{\partial \eta}{\partial z}\right)^2\right\}^{-1/2} \left(\mathbf{e}_r - \frac{\partial \eta}{\partial z}\mathbf{e}_z\right), \qquad (2)$$

where $\mathbf{e}_r$ and $\mathbf{e}_z$ are unit vectors in r and z directions, respectively. We assume that fluid velocity is irrotational in the region so that velocity potentials are $\phi^{(1)}$ and $\phi^{(2)}$ for fluid phases 1 and 2. In each fluid phase

$$\nabla^2 \phi^{(j)} = 0, \quad (j = 1, 2). \qquad (3)$$

The two fluids are subject to an external magnetic field H_1 and H_2 acting along r axis, *i.e.*

$$\mathbf{H}_j = \frac{1}{r} H_j \mathbf{e}_r \quad (j = 1, 2). \qquad (4)$$

Since we assume that there are no free currents at the two phases in the equilibrium state, we find that the magnetic induction is continuous at the interface, *i.e.*

$$\mu_1 H_1 = \mu_2 H_2. \qquad (5)$$

We introduce the magnetic potential $\psi^{(1)}$ and $\psi^{(2)}$ such that

$$\mathbf{h}^{(j)} = -\nabla \psi^{(j)} \quad (j = 1, 2). \qquad (6)$$

Therefore the differential equation satisfied by $\psi^{(j)} (j = 1, 2)$ is Laplace's equation

$$\nabla^2 \psi^{(j)} = 0, \quad (j = 1, 2). \qquad (7)$$

The solutions for $\phi^{(j)}, \psi^{(j)} (j = 1, 2)$ have to satisfy the boundary conditions. The relevant boundary conditions for our configuration are:

(i) On the rigid boundaries $r = a^{(1)}$ and $r = a^{(2)}$:

(1) The normal field velocities vanish on both central solid core and the outer bounding surface.

$$\frac{\partial \phi^{(1)}}{\partial r} = 0 \quad \text{on} \quad r = a^{(1)}, \qquad (8)$$

$$\frac{\partial \phi^{(2)}}{\partial r} = 0 \quad \text{on} \quad r = a^{(2)}; \qquad (9)$$

(2) Tangential components of the magnetic field vanish on these boundaries, *i.e.*,

$$\frac{\partial \psi^{(1)}}{\partial z} = 0 \quad \text{on} \quad r = a^{(1)}, \qquad (10)$$

$$\frac{\partial \psi^{(2)}}{\partial z} = 0 \quad \text{on} \quad r = a^{(2)}. \qquad (11)$$

(ii) On the interface $r = R + \eta(z, t)$:

(1) The tangential components of the magnetic field are continuous at the interface, *i.e.*, $h_{1t} = h_{2t}$, therefore

$$\frac{\partial \eta}{\partial z} \left[\!\left[\frac{\partial \psi}{\partial r} \right]\!\right] + \left[\!\left[\frac{\partial \psi}{\partial z} \right]\!\right] = 0, \qquad (12)$$

where $[\![\ ]\!]$ represents the difference in a quantity as we cross the interface, *i.e.*, $[\![h]\!] = h^{(2)} - h^{(1)}$, where superscripts refer to upper and lower fluids, respectively.

(2) The normal components of the magnetic induction are continuous at the interface, *i.e.*, $\mu_1 h_{1n} = \mu_2 h_{2n}$, therefore

$$\left[\!\left[\mu \frac{\partial \psi}{\partial r} \right]\!\right] - \frac{\partial \eta}{\partial z} \left[\!\left[\mu \frac{\partial \psi}{\partial z} \right]\!\right] = 0. \qquad (13)$$

(3) The conservation of mass across the interface:

$$\left[\!\left[\rho \left(\frac{\partial F}{\partial t} + \nabla \phi \cdot \nabla F \right) \right]\!\right] = 0,$$

$$\text{or} \quad \left[\!\left[\rho \left(\frac{\partial \phi}{\partial r} - \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial z} \frac{\partial \phi}{\partial z} \right) \right]\!\right] = 0. \qquad (14)$$

(4) The interfacial condition for energy is

$$L\rho^{(1)}\left(\frac{\partial F}{\partial t}+\nabla\phi^{(1)}\cdot\nabla F\right)=S(\eta), \qquad (15)$$

where L is the latent heat released when the fluid is transformed from phase 1 to phase 2. Physically, the left-hand side of (15) represents the latent heat released during the phase transformation, while $S(\eta)$ on the right-hand side of (15) represents the net heat flux, so that the energy will be conserved.

In the equilibrium state, the heat fluxes in the direction of r increasing in the fluid phase 1 and 2 are $-K_1(T_1-T_0)/R\log(a/R)$ and $-K_2(T_0-T_2)/R\log(R/b)$, where K_1 and K_2 are the heat conductivities of the two fluids. As in Elhefnawy [4], we denote

$$S(\eta)=\frac{K_2(T_0-T_2)}{(R+\eta)(\log b-\log(R+\eta))} - \frac{K_1(T_1-T_0)}{(R+\eta)(\log(R+\eta)-\log a)}, \qquad (16)$$

and we expand it about $r=R$ by Taylor's expansion, such as

$$S(\eta)=S(0)+\eta S'(0)+\frac{1}{2}\eta^2S''(0)+\cdots, \qquad (17)$$

and we take $S(0)=0$, so that

$$\frac{K_2(T_0-T_2)}{R\log(b/R)}=\frac{K_1(T_1-T_0)}{R\log(R/a)}=G(\text{say}), \qquad (18)$$

indicating that in equilibrium state the heat fluxes are equal across the interface in the two fluids.

From (1, 15), and (17), we have

$$\rho^{(1)}\left(\frac{\partial\phi^{(1)}}{\partial r}-\frac{\partial\eta}{\partial t}-\frac{\partial\eta}{\partial z}\frac{\partial\phi^{(1)}}{\partial z}\right)=\alpha(\eta+\alpha_2\eta^2+\alpha_3\eta^3), \qquad (19)$$

where

$$\alpha=\frac{G\log(b/a)}{LR\log(b/R)\log(R/a)},$$

$$\alpha_2=\frac{1}{R}\left(-\frac{3}{2}+\frac{1}{\log(b/R)}-\frac{1}{\log(R/a)}\right).$$

(5) The conservation of momentum balance, by taking into account the mass transfer across the interface, is

$$\rho^{(1)}(\nabla\phi^{(1)}\cdot\nabla F)\left(\frac{\partial F}{\partial t}+\nabla\phi^{(1)}\cdot\nabla F\right)= \rho^{(2)}(\nabla\phi^{(2)}\cdot\nabla F)\left(\frac{\partial F}{\partial t}+\nabla\phi^{(2)}\cdot\nabla F\right) + (p_2-p_1+\sigma\nabla\cdot\mathbf{n}-[\![\,\mu\{(n_\alpha h_\alpha)^2-\frac{1}{2}h_\gamma h_\gamma\}]\!])|\nabla F|^2, \qquad (20)$$

where p is the pressure and σ is the surface tension coefficient, respectively.

By eliminating the pressure by Bernoulli's equation we can rewrite the above condition (20) as

$$\left[\!\!\left[\rho\left\{\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2-\left\{1+\left(\frac{\partial\eta}{\partial z}\right)^2\right\}^{-1} \times\left(\frac{\partial\phi}{\partial z}\frac{\partial\eta}{\partial z}-\frac{\partial\phi}{\partial r}\right)\left(\frac{\partial\eta}{\partial t}+\frac{\partial\phi}{\partial z}\frac{\partial\eta}{\partial z}-\frac{\partial\phi}{\partial r}\right)\right\}\right]\!\!\right] =-\sigma\frac{\partial^2\eta}{\partial z^2}\left\{1+\left(\frac{\partial\eta}{\partial z}\right)^2\right\}^{-3/2}+\sigma(R+\eta)^{-1}\left\{1+\left(\frac{\partial\eta}{\partial z}\right)^2\right\}^{-1/2} -\left[\!\!\left[\mu\left(\frac{\partial\phi}{\partial\hat{n}}\right)^2-\frac{1}{2}\mu|\nabla\psi|^2\right]\!\!\right]. \qquad (21)$$

where $\partial/\partial\hat{n}$ denotes the differentiation in the direction normal to the surface $r=R+\eta(z,t)$.

To investigate the nonlinear effects on the stability of the system, we employ the method of multiple scales (Lee [8–11]). Introducing ϵ as a small parameter, and variables

$$\eta=\sum_{n=1}^{3}\epsilon^n\eta_n(z_0,z_1,z_2,t_0,t,t_2)+O(\epsilon^4), \qquad (22)$$

$$\phi^{(j)}=\sum_{n=1}^{3}\epsilon^n\phi_n^{(j)}(r;z_0,z_1,z_2,t_0,t_1,t_2)+O(\epsilon^4),\ (j=1,2) \qquad (23)$$

$$\psi^{(j)}=\sum_{n=0}^{3}\epsilon^n\psi_n^{(j)}(r;z_0,z_1,z_2,t_0,t_1,t_2)+O(\epsilon^4), \quad (j=1,2) \qquad (24)$$

The quantities appearing in the field equations (3) and (7) and the boundary conditions (14, 19), and (21) can now be expressed in Maclaurin series expansion around $r=R$. Then, we use (22, 23) and (24) and equate the coefficients of equal power series in ϵ to obtain the linear and the successive nonlinear partial differential equations of various orders .

3 Linear theory

The zeroth order solution yields

$$\psi_0^{(j)}=-H_j\ln r \quad (j=1,2).$$

When surface is perturbed from the equilibrium $r=R$ to $r=R+\eta\exp(\mathrm{i}\theta)$, the linear wave solutions of (2.3) subject to boundary conditions yield

$$\eta_1 = A(z_1, z_2, t_1, t_2)\mathrm{e}^{\mathrm{i}\theta} + \bar{A}(z_1, z_2, t_1, t_2)\mathrm{e}^{-\mathrm{i}\theta}, \tag{25}$$

$$\phi_1^{(j)} = \frac{1}{k}\left(\frac{\alpha}{\rho^{(j)}} - \mathrm{i}\omega\right)A(z_1, z_2, t_1, t_2)E^{(j)}(k, r)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{26}$$

$$\psi_1^{(j)} = \frac{H_j}{R}N_1(1-\nu)A(z_1, z_2, t_1, t_2)F^{(j)}(k, r)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \quad (j = 1, 2) \tag{27}$$

where

$$E^{(i)}(k, r) = \frac{I_0(kr)K_1(ka^{(i)}) + I_1(ka^{(i)})K_0(kr)}{I_1(kR)K_1(ka^{(i)}) - I_1(ka^{(i)})K_1(kR)}, \tag{28}$$

$$F^{(i)}(k, r) = \frac{I_0(kr)K_0(ka^{(i)}) - I_0(ka^{(i)})K_0(kr)}{I_1(kR)K_0(ka^{(i)}) + I_0(ka^{(i)})K_1(kR)}, \tag{29}$$

$$N_1^{-1} = F^{(1)}(k, R) - \nu F^{(2)}(k, R)$$

$$\theta = kz_0 - \omega t_0, \qquad \nu = \mu_1/\mu_2$$

with I_m and $K_m (m = 0, 1)$ are the modified Bessel functions of the first and second kinds, respectively. Here $\bar{A}$ denotes the complex conjugate of amplitude A, and k and ω stand for the wave number and the frequency of the wave.

Substituting (25–27) into the first order terms in (21), we obtain the following dispersion relation

$$D(\omega, k) = -a_0\omega^2 - \mathrm{i}a_1\omega + a_2 = 0, \tag{30}$$

where

$$a_0 = \rho^{(1)}E^{(1)}(k, R) - \rho^{(2)}E^{(2)}(k, R),$$
$$a_1 = \alpha\left\{E^{(1)}(k, R) - E^{(2)}(k, R)\right\},$$
$$a_2 = \frac{\sigma k}{R^2}\left[(R^2k^2 - 1) + \Gamma\left(1 - \frac{1}{\nu}\right)\{1 + N_1(1-\nu)Rk\}\right],$$

where

$$\Gamma = \frac{\mu_2 H_2^2}{R\sigma}.$$

From the properties of Bessel functions, and since α is always positive, we notice that $a_0 > 0$, and $a_1 > 0$. Applying the Routh-Hurwitz criteria to (30), the condition for stability is $a_2 > 0$, which reduces to

$$\Gamma \geq \left(1 - R^2k^2\right)/(1 - 1/\nu)\{1 + N_1(1-\nu)Rk\}. \tag{31}$$

From (31) we see that magnetic field has a stabilizing influence on the wave motion. It is also clear that the mass and heat transfer coefficient α has no effect on the stability condition.

For values of $k \geq k_c$, where

$$\Gamma = \left(1 - R^2k_c^2\right)/(1 - 1/\nu)\{1 + N_1(1-\nu)Rk_c\}. \tag{32}$$

the system is linearly stable. For $k < k_c$ the system is unstable. The corresponding critical frequency, ω_c is zero for this case.

4 Second order solutions

With the use of the first order solutions, we obtained the equations for the second order problem

$$\nabla_0^2\phi_2^{(j)} = -2\frac{\partial^2\phi_2^{(j)}}{\partial z_0 \partial z_1}, \qquad (j = 1, 2) \tag{33}$$

$$\nabla_0^2\psi_2^{(j)} = -2\frac{\partial^2\psi_2^{(j)}}{\partial z_0 \partial z_1}, \qquad (j = 1, 2) \tag{34}$$

where the linear operator is ∇_0^2 is defined to be

$$\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_0^2}, \tag{35}$$

and the boundary conditions at $r = R$.

$$\rho^{(j)}\left\{\frac{\partial\phi_2^{(j)}}{\partial r} - \frac{\partial\eta_2}{\partial t_0}\right\} - \alpha\eta_2 = \left[\rho^{(j)}\left\{\frac{\alpha}{\rho^{(j)}} - \mathrm{i}\omega\right\}\left\{\frac{1}{R} - 2kE^{(j)}\right\} + \alpha\alpha_2\right]A^2\mathrm{e}^{2\mathrm{i}\theta} + \rho^{(j)}\frac{\partial A}{\partial t_1}\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} + 2\alpha\left(\frac{1}{R} + \alpha_2\right)|A|^2, \quad (j = 1, 2) \tag{36}$$

$$\rho^{(2)}\frac{\partial\phi_2^{(2)}}{\partial r} - \rho^{(1)}\frac{\partial\phi_2^{(1)}}{\partial r} - \{\rho^{(2)} - \rho^{(1)}\}\frac{\partial\eta_2}{\partial t_0} = \left[\!\left[\rho\left(\frac{\alpha}{\rho} - \mathrm{i}\omega\right)\left\{\frac{1}{R} - 2kE\right\}\right]\!\right]A^2\mathrm{e}^{2\mathrm{i}\theta} + \{\rho^{(2)} - \rho^{(1)}\}\frac{\partial A}{\partial t_1}\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} \tag{37}$$

$$\frac{\partial\psi_2^{(2)}}{\partial z} - \frac{\partial\psi_2^{(1)}}{\partial z} - \frac{H_2 - H_1}{R}\frac{\partial\eta_2}{\partial z} = \mathrm{i}\frac{H_2k(1-\nu)}{R\nu}\left\{2N_1k(1-\nu) + \frac{1}{R}\right\}A^2\mathrm{e}^{2\mathrm{i}\theta} + \frac{\partial A}{\partial z_1}\frac{H_2(1-\nu)}{R\nu}(1 + F^{(1)} - \nu F^{(2)})\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{38}$$

$$\mu_2\frac{\partial\psi_2^{(2)}}{\partial r} - \mu_1\frac{\partial\psi_2^{(1)}}{\partial r} = -\frac{N_1}{R}\mu_2 H_2 2k^2(1-\nu)[\![F]\!]A^2\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}, \tag{39}$$

$$\rho^{(2)}\frac{\partial\phi_2^{(2)}}{\partial t_0}-\rho^{(1)}\frac{\partial\phi_2^{(1)}}{\partial t_0}+\sigma\left(\frac{\partial^2\eta_2}{\partial z_0^2}+\frac{\eta_2}{R^2}\right)-\frac{\mu_2H_2}{R}\frac{\partial\psi_2^{(2)}}{\partial r}+\frac{\mu_1H_1}{R}\frac{\partial\psi_2^{(1)}}{\partial r}$$
$$=\left\{-\frac{\omega^2}{2}\left[\!\left[\rho\left\{E^2-3\right\}\right]\!\right]+\frac{\alpha^2}{2}\left[\!\left[\frac{1+E^2}{\rho}\right]\!\right]-\mathrm{i}\alpha\omega\left[\!\left[E^2\right]\!\right]+\frac{\sigma}{2R^3}\left(y^2+2\right)+\frac{\Gamma\sigma(1-\nu)}{R^3\nu}\left(\frac{1}{2}N_0^2y^2+2N_0y+\frac{3}{2}-2y^2+\frac{1}{2}N_0N_1y^2\mu_1\left[\!\left[F^2/\mu\right]\!\right]\right)\right\}A^2\mathrm{e}^{2\mathrm{i}\theta}-\left[\!\left[\frac{\rho}{k}\left(\frac{\alpha}{\rho}-\mathrm{i}\omega\right)E\right]\!\right]\frac{\partial A}{\partial t_1}\mathrm{e}^{\mathrm{i}\theta}+\mathrm{c.c.}+\mathrm{NSPT},\tag{40}$$

where $N_0=N_1(1-\nu)$, $E^{(j)}=E^{(j)}(k,R)$, $F^{(j)}=F^{(j)}(k,R)$, $(j=1,2)$, $y=Rk$, and NSPT is the nonsingular producing term.

The non secularity conditions for the existence of the uniformly valid solution are

$$\frac{\partial A}{\partial t_1}+V_\mathrm{g}\frac{\partial A}{\partial z_1}=0,\tag{41}$$

and its complex conjugate relation and V_g is the group velocity of the wave

$$V_\mathrm{g}=\frac{\mathrm{d}\omega}{\mathrm{d}k}\,.\tag{42}$$

Equations (33) to (40) furnish the second order solutions:

$$\eta_2=-2\left(\frac{1}{R}+\alpha_2\right)|A|^2+A_2\mathrm{e}^{2\mathrm{i}\theta}+\bar{A}_2\mathrm{e}^{-2\mathrm{i}\theta},\tag{43}$$

$$\phi_2^{(j)}=\left[-\frac{\mathrm{i}}{k}\left\{\frac{\alpha}{\rho^{(j)}}-\mathrm{i}\omega\right\}\left\{rL^{(j)}(k,r)+a^{(j)}S^{(j)}(k,r)-\left[RE^{(j)}-a^{(j)}M^{(j)}(k,R)\right]E^{(j)}(k,r)\right\}\frac{\partial A}{\partial z_1}+\frac{1}{k}\frac{\partial A}{\partial t_1}E^{(j)}(k,r)\right]\mathrm{e}^{\mathrm{i}\theta}+B^{(j)}\mathrm{e}^{2\mathrm{i}\theta}E^{(j)}(2k,r)+\mathrm{c.c.}+b^{(j)}(t_0,t_1,t_2),\quad(j=1,2)\tag{44}$$

$$\psi_2^{(j)}=\frac{H_2}{R^2}\left(\frac{1}{\nu}-1\right)C^{(j)}A^2\mathrm{e}^{2\mathrm{i}\theta}F^{(j)}(2k,r)-\mathrm{i}(1-\nu)\frac{N_1H_j}{R}\left[rT^{(j)}(k,r)-a^{(j)}P^{(j)}(k,r)-D^{(j)}F^{(j)}(k,r)\right]\frac{\partial A}{\partial z_1}\mathrm{e}^{\mathrm{i}\theta}+\mathrm{c.c.},\quad(j=1,2).\tag{45}$$

Symbols in the above equations are found in the Appendix. Equation (41) shows that, to the lowest order in ϵ, the amplitude A is constant in a frame of reference moving with the group velocity of the waves.

Furthermore, we have assumed the $D(2\omega,2k)\neq 0$. The case when $D(2\omega,2k)=0$ corresponds to the second harmonic resonance.

5 Third order solutions

We examine now the third order problem:

$$\nabla_0^2\phi_3^{(i)}=-\frac{\partial^2\phi_1^{(i)}}{\partial z_1^2}-2\frac{\partial^2\phi_1^{(i)}}{\partial z_0\partial z_2}-2\frac{\partial^2\phi_2^{(i)}}{\partial z_0\partial z_1},\qquad(i=1,2)\tag{46}$$

$$\nabla_0^2\psi_3^{(i)}=-\frac{\partial^2\psi_1^{(i)}}{\partial z_1^2}-2\frac{\partial^2\psi_1^{(i)}}{\partial z_0\partial z_2}-2\frac{\partial^2\psi_2^{(i)}}{\partial z_0\partial z_1},\qquad(i=1,2).\tag{47}$$

Substituting the values of $\phi_1^{(i)}$ from (26) and $\phi_2^{(i)}$ from (44) into (46), we obtain $\phi_3^{(j)}(k,r),(j=1,2)$ which are listed in the Appendix. The solution of (47) is also given in the Appendix.

With the third order solution the condition for third order perturbation to be nonsecular is

$$\mathrm{i}\left(\frac{\partial A}{\partial t_2}+V_\mathrm{g}\frac{\partial A}{\partial z_2}\right)+P\frac{\partial^2A}{\partial z_1^2}=QA^2\bar{A}+RA,\tag{48}$$

where

$$P=\frac{1}{2}\frac{\mathrm{d}V_\mathrm{g}}{\mathrm{d}k},$$

$$R=2\sigma\hat{\mu}k_ck\left(\frac{\partial D}{\partial\omega}\right)^{-1},$$

where $\hat{\mu}$ is defined by $k=k_c+\hat{\mu}\epsilon^2$ with k_c equal to critical wave number.

It is now appropriate to introduce the transformations

$$\zeta=\epsilon^{-1}(z_2-V_\mathrm{g}t_2)=(z_1-V_\mathrm{g}t_1)=\epsilon(z-V_\mathrm{g}t)\quad\text{and}\quad\tau=t_2=\epsilon t_1=\epsilon^2t.$$

Equation (48) is reduced to

$$\mathrm{i}\frac{\partial A}{\partial\tau}+P\frac{\partial^2A}{\partial\zeta^2}=QA^2\bar{A}+RA,\tag{49}$$

which is a complex Ginzburg-Landau equation, *i.e.*

$$P=P_r+\mathrm{i}P_i,\quad Q=Q_r+\mathrm{i}Q_i,\quad\text{and}\quad R=R_r+\mathrm{i}R_i.$$

The stability of a Ginzburg-Landau equation (49) is discussed by Lange and Newell [12], and Matkowsky and Volpert [13]. They showed that stability conditions are

$$P_rQ_r+P_iQ_i>0\quad\text{and}\quad Q_i<0,\tag{50}$$

provided that $R_r=0$.

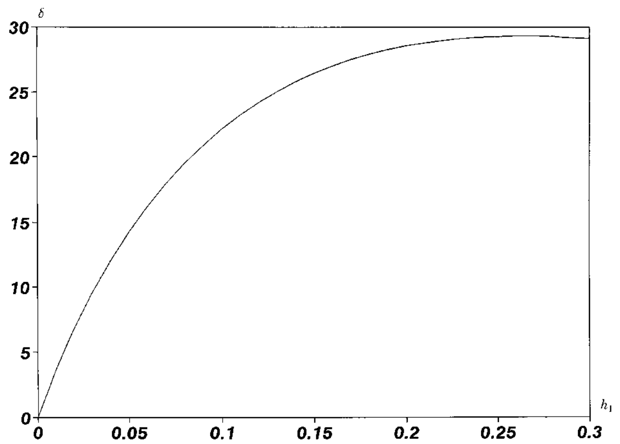

Fig. 2. Variation of δ with respect to h_1.

We notice that the condition $R_r = 0$ is satisfied when $\omega = 0$, and $P_r = Q_r = 0$. In this case, (49) reduces to the nonlinear diffusion equation,

$$\mathrm{i}\frac{\partial A}{\partial \tau} + P_i\frac{\partial^2 A}{\partial \zeta^2} = Q_i A^2 \bar{A} + R_i A, \tag{51}$$

where

$$Q_i = \frac{k}{a_1}\left\{\alpha^2\left[\!\left[\left(\frac{N+1}{R}+\alpha_2-2kE\right)\left\{\frac{E(2k,R)E-1}{\rho}\right\}+\frac{3}{R\rho}\right]\!\right]\right.$$
$$\left.+\frac{\sigma}{R^4}\left[2N(k^2R^2-1)+4R\alpha_2+7-\frac{1}{2}k^2R^2(1-3k^2R^2)+\Gamma B_0\right]\right\}. \tag{52}$$

with the symbols explained in the Appendix.

The solution of the nonlinear diffusion equation (51) is valid near the marginal state (*i.e.* $\omega = 0$) and therefore be used to study the stability of the system. From inequalities (50), we find the stability conditions of (51) are

$$P_i < 0 \quad \text{and} \quad Q_i < 0. \tag{53}$$

The stability depends on the thickness of vapor h and α. The stability can therefore be discussed by dividing the $h-k-$plane into stable and unstable regions. The transition curves are given by the vanishing of P_i and Q_i.

6 Numerical results

In expression (74), k and k_c are essentially the same. From (52) and criteria (53), it is seen that a relevant nondimensional parameter is δ which is defined in (A.21) and from (52) we can obtain the value δ for which the system is stable. The stability depends on various parameters. Let h_1 denote thickness of the vapor. In Figure 2, we show the variation of δ with respect to the thickness of the vapor. Here we have chosen $a^{(1)} = 1$ cm, and $a^{(2)} = 2$ cm, and $\Gamma = 2$, $\mu_1 = 1.007$, $\mu_2 = 1.7$, $\rho^{(1)} = 3.652 \times 10^{-4}$ gm cm^{-3}, $\rho^{(2)} = 5.97 \times 10^{-2}$ gm cm^{-3}. The region above the curve is the stable region, while below the curve is the region of instability. From this figure we see that as h_1 increases δ increases also. This means, since α is proportional to the heat flux of the system, with the same heat flux, the thinner the vapor the easier the system can be stabilized.

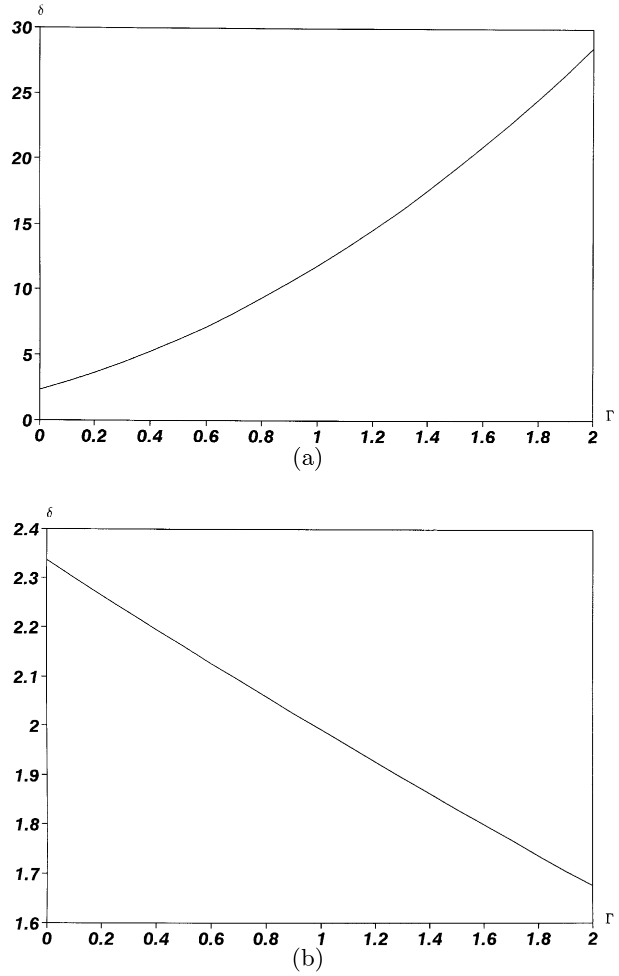

Fig. 3. Variation of δ with respect to magnetic strength.

In Figure 3a, b we show the variation of δ with respect to Γ. In Figure 3a we chose $a^{(1)}, a^{(2)}$, ν, as in Figure 2, and $h_1 = 0.2$ cm. As in Figure 2, the region above the curve is the stable region, and below the curve is the unstable region. Here we see that as Γ increases, δ increases. This means that the stronger magnetic field has destabilizing effect. On the other hand, in Figure 3b, $\nu = 1.1$ is chosen while other values remain unchanged. In this case contrary to Figure 3a, we notice that stronger magnetic field has stabilizing effect.

In Figure 4, the variation of δ with respect to ν is shown. Here, $\rho^{(1)}$, $\rho^{(2)}, h_1$ are as in Figure 3a, but $\Gamma = 2$. This figure shows that as ν is increased, the system is stabilized.

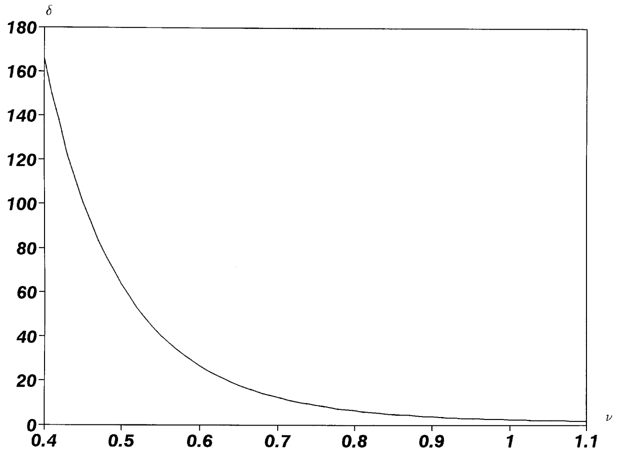

Fig. 4. Variation of δ with respect to ν.

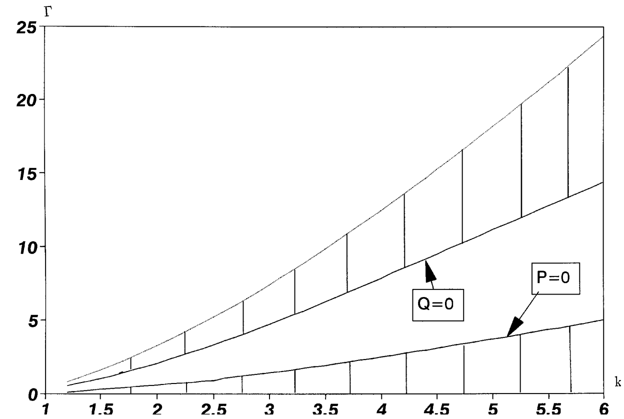

Fig. 5. Stability diagram for the system as considered in Figure 2 ($h = 0.2$ cm).

7 Stability analysis when $\alpha = 0$

When the mass and heat transfer is neglible, the analysis can be simplified by taking $\alpha = 0$ in the evolution equation (48). In this case, the imaginary parts of P, Q, and R in (49) are equal to zero. Therefore (49) is now the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial A}{\partial \tau} + P_r \frac{\partial^2 A}{\partial \zeta^2} = Q_r A^2 \bar{A} + R_r A, \qquad (54)$$

where P_r, Q_r and R_r are the values of P, Q, and R, respectively, when $\alpha = 0$.

It is known that the solutions of (54) are unstable against modulation if

$$P_r Q_r < 0. \qquad (55)$$

The modulational instability is characterized by the criterion (55) which yields the value of k at which the instability occurs. Such criterion depends upon the wave number, and the thickness of the fluid h_1 or h_2.

In Figure 5 we have sketched the transition curves across which P and Q change signs for different values of Γ when $h_1 = 0.2$ cm. The values of $\rho^{(1),(2)}, \mu_{1,2}, \sigma$ are as in Figure 2. Here, the shaded region is the region of stability where both P and Q are both negative or positive. The dotted line is the linear curve. The curve representing second harmonic resonance can not be seen in this diagram since $\rho^{(1)} < \rho^{(2)}$. It is interesting to observe from Figure 5 that as k increases, the region of stability is enlarged. Figure 6 represents the same system as considered in Figure 5, but in this case $h = 0.4$ cm is chosen. The situation is very similar to Figure 5, except that we can notice that for the same value of k, the transition curves are determined at higher values of Γ than in the case of Figure 5.

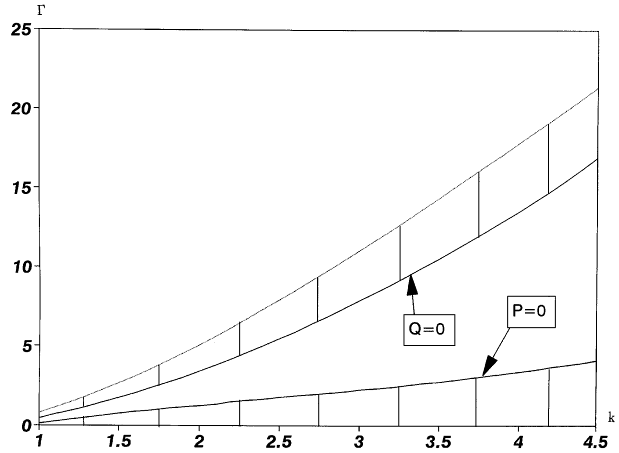

Fig. 6. Stability diagram for the system as considered in Figure 2 ($h = 0.4$ cm).

Conclusion

In the present paper, nonlinear analysis of Rayleigh-Taylor instability across the cylindrical interface of magntic fluids with heat and mass transfer by using the multiple-scale expansion method is presented. A Ginzburg-Landau equation describing the evolution of nonlinear waves is obtained. Also a nonlinear Schrödinger equation is obtained when heat and mass transfer is absent. In this solution, the effect of heat and mass transfer is measured by parameters α, α_2, and Γ. It is found that when α is large enough or when the nondimensional parameter δ is large enough, the system, which would be unstable classically, can be stabilized for finite amplitude disturbances. Since α is proportional to the heat flux of the system, when the heat flux is sufficiently intense, the system can be stabilized. The important point is that the nonlinear effect can indeed increase the stability range of the Rayleigh-Taylor system when there is strong heat and mass transfer while this is not the case for the ordinary Rayleigh-Taylor instability. We also found that with the stronger magnetic field the system is more stable when $\nu = 1.1$. Unlike linear theory, with nonlinear theory, it is evident that the mass and heat transfer plays an important role in the stability of fluid, in a situation like film boiling.

Appendix

$$A_2 = -\frac{2k}{D(2\omega,2k)}\Bigg\{\Bigg[\!\!\Bigg[\frac{\rho \mathrm{i}\omega}{k}E(2k,R)\beta + \frac{\rho}{2}E^2 \times \left(\frac{\alpha}{\rho}-\mathrm{i}\omega\right)^2 + \frac{3\omega^2\rho^2+\alpha^2}{2\rho}\Bigg]\!\!\Bigg] + \frac{\sigma}{R^3}\Bigg[1+\frac{y^2}{2}+\Gamma\left(\frac{1}{\nu}-1\right)\Bigg(\frac{1}{2}N_0^2y^2+2N_0y+\frac{3}{2}-2y^2+N_2(1-\nu)y + \frac{1}{2}N_0N_1y^2\mu_1[\![F^2/\mu]\!] + 2y^2N_1N_2\{(1-\nu)^2 + \nu[\![F]\!][\![F_2]\!]\}\Bigg)\Bigg]\Bigg\}A^2, \quad (A.1)$$

$$B^{(j)} = \frac{1}{2k}\left[\beta^{(j)}A^2 + \left\{\frac{\alpha}{\rho^{(j)}} - 2\omega\mathrm{i}\right\}A_2\right], \quad (A.2)$$

$$C^{(1)} = N_2\left[A_2 - \frac{1}{2} - yN_1\left\{1-\nu+\nu[\![F]\!]F_2^{(2)}\right\}\right], \quad (A.3)$$

$$C^{(2)} = \nu N_2\left[A_2 - \frac{1}{2} - yN_1\left\{1-\nu+[\![F]\!]F_2^{(1)}\right\}\right], \quad (A.4)$$

$$D^{(1)} = N_1\left\{\nu[\![RF+aQ]\!]F^{(2)} + (1-\nu)R + \mu_1[\![aP/\mu]\!]\right\}, \quad (A.5)$$

$$D^{(2)} = N_1\left\{[\![RF+aQ]\!]F^{(1)} + (1-\nu)R + \mu_1[\![aP/\mu]\!]\right\}, \quad (A.6)$$

$$F_2^{(j)} = F^{(j)}(2k,R), (j=1,2) \quad N_2^{-1} = F_2^{(1)} - \nu F_2^{(2)}, \quad (A.7)$$

$$\beta^{(j)} = k\left\{\frac{\alpha}{\rho^{(j)}} - \mathrm{i}\omega\right\}\left\{\frac{1}{y} - 2E^{(j)}\right\} + \frac{\alpha\alpha_2}{\rho^{(j)}}, \quad (A.8)$$

$$S^{(i)}(k,r) = \frac{1}{\gamma^{(i)}}\left\{K_0(kr)I_0\left(ka^{(i)}\right) - I_0(kr)K_0\left(ka^{(i)}\right)\right\}, \quad (A.9)$$

$$P^{(i)}(k,r) = \frac{1}{\delta^{(i)}}\left\{K_0(kr)I_1\left(ka^{(i)}\right) + I_0(kr)K_1\left(ka^{(i)}\right)\right\}, \quad (A.10)$$

$$L^{(i)}(k,r) = \frac{f^{(i)}(k,r)}{\gamma^{(i)}}, \quad M^{(i)}(k,r) = \frac{g^{(i)}(k,r)}{\gamma^{(i)}}, \quad (A.11)$$

$$Q^{(i)}(k,r) = -\frac{f^{(i)}(k,r)}{\delta^{(i)}}, \quad T^{(i)}(k,r) = \frac{g^{(i)}(k,r)}{\delta^{(i)}}, \quad (A.12)$$

$$f^{(i)}(k,r) = I_1(kr)K_1\left(ka^{(i)}\right) - K_1(kr)I_1\left(ka^{(i)}\right), \quad (A.13)$$

$$g^{(i)}(k,r) = K_1(kr)I_0\left(ka^{(i)}\right) + I_1(kr)K_0\left(ka^{(i)}\right), \quad (A.14)$$

$$\gamma^{(i)} = f^{(i)}(k,R), \quad \delta^{(i)} = g^{(i)}(k,R), \quad (A.15)$$

$$P^{(i)} = P^{(i)}(k,R), \quad Q^{(i)} = Q^{(i)}(k,R), \quad (A.16)$$

$$\phi_3^{(j)} = -\frac{1}{k}\left(\frac{\alpha}{\rho^{(j)}} - \mathrm{i}\omega\right)\Bigg[\frac{1}{2}\Bigg\{r^2E^{(j)}(k,r) - \frac{r}{k}L^{(j)}(k,r)\Bigg\} - arM^{(j)}(k,r) - \{RE-aM\}rL^{(j)}(k,r) + \frac{G^{(j)}(k,r)}{k\gamma^{(j)}}\left\{\frac{1}{2k} - RE + aM\right\} - \left\{\frac{R}{2}(E+y) + y\left[aF - (RE-aM)\right]E\right\}\frac{E^{(j)}(k,r)}{k}\Bigg]\frac{\partial^2 A}{\partial z_1^2}\mathrm{e}^{\mathrm{i}\theta} - \frac{\mathrm{i}}{k}\left\{rL^{(j)}(k,r) + aS^{(j)}(k,r) - (RE-aM)E^{(j)}(k,r)\right\} \times \left[\left\{\frac{\alpha}{\rho^{(j)}} - \mathrm{i}\omega\right\}\frac{\partial A}{\partial z_2} + \frac{\partial^2 A}{\partial z_1 \partial t_1}\right]\mathrm{e}^{\mathrm{i}\theta} + \frac{\partial A}{\partial t_2}\frac{E^{(j)}(k,r)}{k}\mathrm{e}^{\mathrm{i}\theta} + \tilde{\phi}_3^{(j)}, \quad (A.17)$$

where

$$G^{(i)}(k,r) = \frac{1}{\gamma^{(i)}}\{K_0(kr)I_1(kR) + I_0(kr)K_1(kR)\},$$

and, for brevity of notations, we used

$$E = E^{(j)}(k,R), \quad M = M^{(j)}(kR), \quad F = F^{(j)}(kR), \quad a = a^{(j)}$$

and

$$\tilde{\phi}_3^{(j)} = -kE^{(j)}(k,r)\Bigg[2\left\{E^{(j)}(2k,R) - \frac{1}{y}\right\}B^{(j)} + \Bigg\{-2\left(E - \frac{1}{y}\right)\left(\frac{\alpha}{\rho^{(j)}} - \mathrm{i}\omega\right) \times \left(\frac{1}{y} + \frac{\alpha_2}{k}\right) + \frac{1}{2}\left(1 + \frac{2}{y^2} - \frac{E}{y}\right)\left(\frac{3\alpha}{\rho^{(j)}} - \mathrm{i}\omega\right) - \frac{\alpha}{\rho^{(j)}} - \mathrm{i}\omega + \frac{\alpha}{\rho^{(j)}}\left(\frac{4\alpha_2}{k}\left\{\frac{1}{y} + \frac{\alpha_2}{k}\right\} - \frac{3\alpha_3}{k^2}\right)\Bigg\}A^2 - \left\{\left(\frac{\alpha}{\rho^{(j)}} + \mathrm{i}\omega\right)\left(E + \frac{1}{y}\right) + \frac{2\alpha\alpha_2}{\rho^{(j)}k}\right\}\frac{A_2}{k}\Bigg]\bar{A}\mathrm{e}^{\mathrm{i}\theta} + H_1I_0(2kr)\mathrm{e}^{2\mathrm{i}\theta} + J_1I_0(3kr)\mathrm{e}^{3\mathrm{i}\theta} + \mathrm{c.c.}, \quad (A.18)$$

where the arbitrary functions H_1 and J_1 can be determined from boundary conditions.

$$\psi_3^{(j)} = \frac{N_1}{R} H_j (1-\nu) \Bigg[\Bigg\{ rT^{(j)}(k,r) - a^{(j)} P^{(j)}(k,r) \Bigg\} \times \left\{ \left(\frac{1}{2k} + D^{(j)} \right) \frac{\partial^2 A}{\partial z_1^2} - \mathrm{i} \frac{\partial A}{\partial z_2} \right\} - \left\{ \frac{1}{2} r^2 F^{(j)}(k,r) + a^{(j)} r Q^{(j)}(k,r) \right\} \frac{\partial^2 A}{\partial z_1^2} \Bigg] \times \mathrm{e}^{\mathrm{i}\theta} + d^{(j)} F^{(j)}(k,r) \mathrm{e}^{\mathrm{i}\theta}, \quad (j=1,2) \quad \text{(A.19)}$$

where $d^{(j)}$ is to be determined by boundary conditions.

$$N = -\frac{1}{4y^2 - 1 + \Gamma(1-\frac{1}{\nu})(N_2 2y(1-\nu)+1)} \left[1 + \frac{y^2}{2} + \Gamma \left(\frac{1}{\nu} - 1 \right) \times \left(\frac{1}{2} N_0^2 y^2 + 2N_0 y + \frac{3}{2} - 2y^2 + N_2(1-\nu) y + \frac{1}{2} N_0 N_1 y^2 \mu_1 [\![F^2/\mu]\!] + 2y^2 N_1 N_2 \left\{ (1-\nu)^2 + \nu [\![F]\!][\![F_2]\!] \right\} \right) - \frac{1}{2} \delta \left\{ 1 + E_1^2 - \hat{\rho} \left(1 + E_2^2 \right) \right\} \right], \quad \text{(A.20)}$$

where

$$\delta = \frac{\alpha^2 R^3}{\rho^{(1)} \sigma}, \quad \hat{\rho} = \frac{\rho^{(1)}}{\rho^{(2)}}, \quad \text{(A.21)}$$

$$B_0 = \left(\frac{1}{\nu} - 1 \right) \Bigg[-(1+yN_0) \Bigg\{ 2(N+c_0) - 6 + \frac{5}{2} y^2 - 3yN_0 - 4y [\![C]\!] \Bigg\} - (N+c_0)\left(1-y^2\right) - 2y^2 - 2N_0 y^2 [\![CFF_2]\!] + yN_1 \Bigg(\Bigg\{ -NN_1 y(1-\nu) - N + 1 - \frac{3}{2} y^2 + \frac{3}{2} N_1 y (1-\nu) \Bigg\} (1-\nu) - y \Bigg\{ \frac{2}{\mu_2} \left[\!\!\left[\mu C \left(F_2 - \frac{1}{y} \right) \right]\!\!\right] - \nu N_1 [\![F]\!] \left(N + \frac{3}{2} \right) \Bigg\} [\![F]\!] \Bigg) \Bigg], \quad \text{(A.22)}$$

where $\quad c_0 = -2(1 + R\alpha_2)$.

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